

MTH250 - CALCULUS

Lecture No. 28

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These lecture notes are prepared to serve as handouts for the video lectures on the course *MTH-250 Calculus* of Virtual Campus, COMSATS Institute of Information Technology, Islamabad, Pakistan. The notes were derived from an extensive collection of notes and books. Most of the theoretical details as well as worked problems are from *Calculus*, by *H. Anton, I Bivens and S. Davis*, John Wiley & Sons, 10th edition, 2012.

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LECTURE 28

Line integrals I

28.1 Introduction to line integrals

We consider the problem of finding the mass of a very thin wire whose linear density function (mass per unit length) is known.

We model the wire by a smooth curve C between two points P and Q in $3D$; see Figure 28.1.



Figure 28.1: A wire modeled a curve C between points P and Q .

Let $f(x, y, z)$ denote the corresponding value of the density function at a point (x, y, z) on the curve in space.

In order to find the mass of the wire, we proceed as follows

- Divide C into n small sections using succession of distinct partition points

$$P = P_0, P_1, P_2, \dots, P_{n-1}, P_n = Q.$$

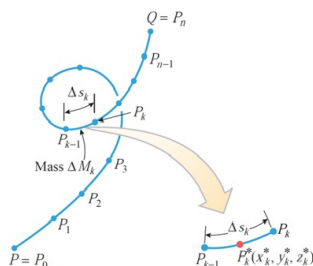


Figure 28.2: Partitioning of the wire into n small arcs.

- Let ΔM_k denote the Mass of the k^{th} section and Δs_k be the arc length of it.
- Choose arbitrary sampling point $P_k^*(x_k^*, y_k^*, z_k^*)$ on the k^{th} arc.
- Then $\Delta M_k \simeq f(x_k^*, y_k^*, z_k^*) \Delta s_k$.
- Therefore, the mass M of the entire wire is approximately given by

$$M \simeq \sum_{k=1}^n f(x_k^*, y_k^*, z_k^*) \Delta s_k.$$

Definition 28.1 (Line integral).

If C is a smooth curve, then the line integral of f with respect to s along C is

$$\int_C f(x, y) ds = \lim_{\max \Delta s_k \rightarrow 0} \sum_{k=1}^n f(x_k^*, y_k^*) \Delta s_k$$

or

$$\int_C f(x, y, z) ds = \lim_{\max \Delta s_k \rightarrow 0} \sum_{k=1}^n f(x_k^*, y_k^*, z_k^*) \Delta s_k$$

provided this limit exists and does not depend on the choice of partition or on the choice of sampling points.

We define an integral that is similar to a single integral except that, instead of integrating over an interval $[a, b]$, we integrate over a curve C . Such integrals are called line integrals. Line integrals have numerous applications in fluid flow, electricity, magnetism and medical imaging.

28.1.1 Interpretations

The line integrals can be interpreted as follows

- If $f(x, y, z)$ is the density of a thin wire (represented by) C . Then $\int_C f(x, y, z) ds$ is the mass of the entire wire.

If C is a curve in the xy -plane and $f(x, y)$ is non-negative continuous function defined on C , then $\int_C f(x, y) ds$ can be interpreted as the area A of the sheet that is swept out by a vertical line segment that extends upward from the point (x, y) to a height of $f(x, y)$ and moves along C from one end point to the other *i.e.*

$$A = \lim_{\max \Delta s_k \rightarrow 0} \sum_{k=1}^n f(x_k^*, y_k^*) \Delta s_k = \int_C f(x, y) ds.$$

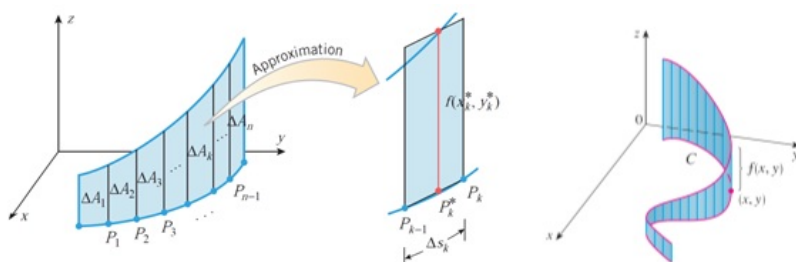


Figure 28.3: Physical interpretations of line integrals.

28.2 Arc length

We start with a plane curve C given by the parametric equations

$$x = x(t), \quad y = y(t), \quad a \leq t \leq b.$$

The direction in which the graph of a pair of parametric equation is traced as the parameter increases is called the orientation imposed on the curve by the equations.

There is a difference between a curve (set of points) and a parametric curve (a curve with an orientation imposed on it by a set of parametric equations).

Definition 28.2 (Arc length for a parametric curve).

If no segment of the curve represented by the parametric equations

$$x = x(t), \quad y = y(t), \quad a \leq t \leq b$$

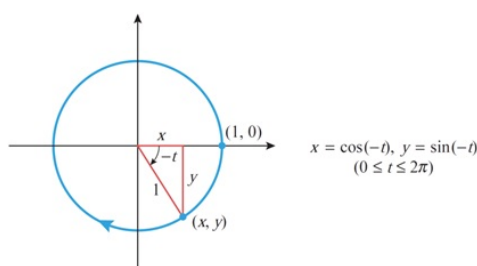


Figure 28.4: Orientation imposed on it by a set of parametric equations.

is traced more than once as t increases from a to b , and if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous functions for $a \leq t \leq b$ then the arc length L of the curve is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Example 28.1. The circumference of the circle parameterized by

$$x = a \cos t \quad y = a \sin t$$

is given by

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{2\pi} \sqrt{(-a \sin t)^2 + (a \cos t)^2} dt = \int_0^{2\pi} a dt = 2\pi a.$$

Definition 28.3 (Arc length for a polar curve).

If no segment of the polar curve $r = f(\theta)$ is traced more than once as θ increases from α to β , and if $\frac{dr}{d\theta}$ is continuous functions for $\alpha \leq \theta \leq \beta$ then the arc length L from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_\alpha^\beta \sqrt{|f(\theta)|^2 + |f'(\theta)|^2} d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Example 28.2. Find the arc length of the spiral $r = e^\theta$ between $\theta = 0$ and $\theta = \pi$.

Solution.

$$L = \int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^\pi \sqrt{(e^\theta)^2 + (e^\theta)^2} d\theta = \int_0^\pi \sqrt{2} e^\theta d\theta = \sqrt{2}(e^\pi - 1).$$

□

A curve C with parametric equations

$$x = x(t), \quad y = y(t)$$

in xy -plane can be given by the vector equation

$$\mathbf{r}(t) = x(t)\hat{i} + y(t)\hat{j}.$$

Then, the arc length is given by

$$L = \int_a^b |\mathbf{r}'(t)| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt.$$

Similarly, a curve C in $3D$ with parametric equations

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

can be given by the vector equation

$$\mathbf{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}.$$

We assume that C is a *smooth curve*, informally, means that, $\mathbf{r}$ is differentiable, $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq 0$.

28.3 Evaluating line integrals

We have already defined the line integrals. However, the definition is not sufficient enough to practically evaluate those line integrals. In this section, we try find out practical way to evaluate line integrals. We proceed as follows.

Let C be a smooth curve in xy -plane given by the vector equation

$$\mathbf{r}(t) = x(t)\hat{i} + y(t)\hat{j}.$$

- Divide the parameter interval $[a, b]$ into n sub-intervals $[t_{i-1}, t_i]$ of equal width.

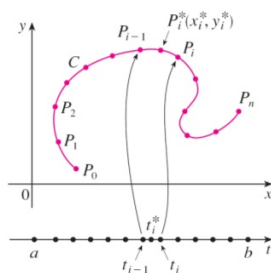


Figure 28.5: Partitioning of a parametric curve.

- We let $x_i = x(t_i)$ and $y_i = y(t_i)$.
- Then, the corresponding points $P_i(x_i, y_i)$ divide C into n sub-arcs with lengths $\Delta s_1, \Delta s_2, \dots, \Delta s_n$

- From the previous section,

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

- Recall from mean value theorem for integrals that as $r(t)$ is smooth, there exists $t_k^* \in [t_{k-1}, t_k]$ such that

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \sqrt{(x'(t))^2 + (y'(t))^2} dt = \Delta t_k \sqrt{(x'(t_k^*))^2 + (y'(t_k^*))^2}$$

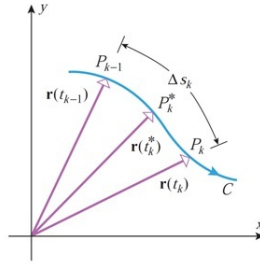


Figure 28.6: Sub-arcs and their arc-lengths.

- Let $x_k^* = x(t_k^*)$ and $y_k^* = y(t_k^*)$.
- Since the parametrization of C is smooth $\Delta s_k \rightarrow 0$ if and only if $\Delta t_k \rightarrow 0$

$$\begin{aligned} \int_C f(x, y) ds &= \lim_{\max \Delta s_k \rightarrow 0} \sum_{k=1}^n f(x_k^*, y_k^*) \Delta s_k \\ &= \lim_{\max \Delta s_k \rightarrow 0} \sum_{k=1}^n f(x(t_k^*), y(t_k^*)) \Delta t_k \sqrt{(x'(t_k^*))^2 + (y'(t_k^*))^2} \\ &= \lim_{\max \Delta t_k \rightarrow 0} \sum_{k=1}^n f(x(t_k^*), y(t_k^*)) \Delta t_k \sqrt{(x'(t_k^*))^2 + (y'(t_k^*))^2} \\ &= \int_a^b f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2} dt. \end{aligned}$$

Theorem 28.4. If a curve C in 2D can be parameterized by equations

$$x = x(t), \quad y = y(t), \quad a \leq t \leq b,$$

then

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

Similarly, if a curve C in 3D can be parameterized by equations

$$x = x(t), \quad y = y(t), \quad z = z(t) \quad a \leq t \leq b,$$

then

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

Example 28.3. Evaluate the line integral $\int_C (xy + z^3) ds$ from $(1, 0, 0)$ to $(-1, 0, \pi)$ along the helix C that is represented by the parametric equations

$$x = \cos t, \quad y = \sin t, \quad z = t, \quad 0 \leq t \leq \pi.$$

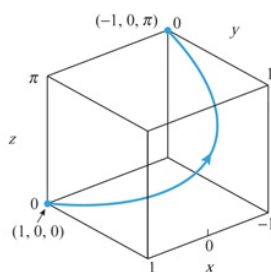


Figure 28.7: The helix C with parametric equations $x = \cos t, y = \sin t, z = t, 0 \leq t \leq \pi$.

Solution.

$$\begin{aligned} \int_C (xy + z^2) ds &= \int_0^\pi (\cos t \sin t + t^3) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \\ &= \int_0^\pi (\cos t \sin t + t^3) \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} dt \\ &= \sqrt{2} \int_0^\pi (\cos t \sin t + t^3) dt \\ &= \sqrt{2} \left[\frac{\sin^2 t}{2} + \frac{t^4}{4} \right]_0^\pi = \frac{\sqrt{2}\pi^4}{4} \end{aligned}$$

.

□

Example 28.4. Find the area of the surface extending upward from the circle $x^2 + y^2 = 1$ in the xy -plane to the parabolic cylinder $z = 1 - x^2$.

Solution. It follows that the area A of the surface can be expressed as the line integral

$$A = \int_C (1 - x^2) ds$$

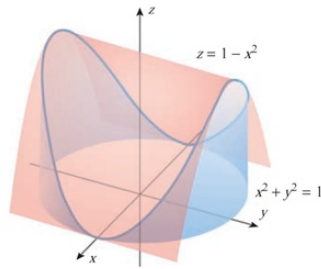


Figure 28.8: Area of the surface extending upward from the circle $x^2 + y^2 = 1$ to the parabolic cylinder $z = 1 - x^2$

where C is the circle $x^2 + y^2 = 1$. This circle can be parametrized in terms of arc length as

$$x = \cos s, \quad y = \sin s, \quad 0 \leq s \leq 2\pi.$$

Thus,

$$\begin{aligned} A &= \int_C (1 - x^2) ds = \int_0^{2\pi} (1 - \cos^2 s) ds \\ &= \int_0^{2\pi} \sin^2 s ds = \frac{1}{2} \int_0^{2\pi} (1 - \cos 2s) ds = \pi \end{aligned}$$

□

Remark 28.5. The value of the line integral does not depend on the parameterization of the curve provided the curve is traversed exactly once as t increases from a to b .

When C is the line segment that joins $(a, 0)$ to $(b, 0)$, using x as the parameter, we can write the parametric equations of C as:

$$x = x, \quad y = 0, \quad a \leq x \leq b.$$

In this case, the line integral reduces to an ordinary single integral, i.e.

$$\int_C f(x, y) ds = \int_a^b f(x, 0) dx.$$

28.3.1 Line integral along piecewise smooth curves

Definition 28.6 (Piecewise smooth curve).

Let C be a union of a finite number of smooth curves $C_1, C_2, \dots, C_n$ where the initial point of C_{i+1} is the terminal point of C_i . Then C is called a piecewise smooth curve.

Then, we define the line integral of f along C as the sum of the integrals of f along each of the smooth pieces of C , that is,

$$\int_C f(x, y, z) ds = \int_{C_1} f(x, y, z) ds + \int_{C_2} f(x, y, z) ds + \dots + \int_{C_n} f(x, y, z) ds$$

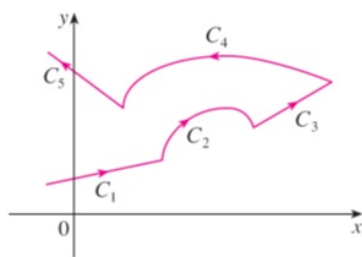
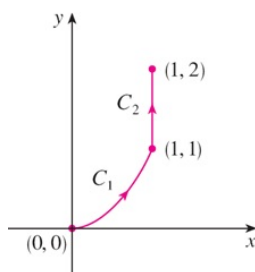


Figure 28.9: A piecewise smooth curve.

Example 28.5. Evaluate $\int_C 2x ds$ where C consists of the arc C_1 of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the vertical line segment C_2 from $(1, 1)$ to $(1, 2)$.

Figure 28.10: The piecewise smooth curve $C = C_1 + C_2$.

Solution. Since C_1 is the graph of a function of x , we can choose x as the parameter. Then, the equations for C_1 become:

$$x = x, \quad y = x^2, \quad 0 \leq x \leq 1.$$

$$\begin{aligned} \int_{C_1} 2x dx &= \int_0^1 2x \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^1 2x \sqrt{1 + 4x^2} dx \\ &= \frac{1}{4} \frac{2}{3} (1 + 4x^2)^{3/2} \Big|_0^1 = \frac{5\sqrt{5} - 1}{6}. \end{aligned}$$

On C_2 , we choose y as the parameter. So, the equations of C_2 are:

$$x = 1, \quad y = y, \quad 1 \leq y \leq 2.$$

Therefore

$$\int_{C_2} 2x ds = \int_1^2 2(1) \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2} dy = \int_1^2 2 dy = 2.$$

Thus,

$$\int_C 2x ds = \int_{C_1} 2x ds + \int_{C_2} 2x ds = \frac{5\sqrt{5}-1}{6} + 2.$$

□

28.4 Line integrals with respect to x, y, z

Definition 28.7. We have second type of line integral in which we replace the ds in the integral by dx , dy or dz . Suppose f is a function defined on a smooth curve C in the xy -plane and that partition points of C are denoted by $P_k(x_k, y_k)$. Letting

$$\Delta x_k = x_k - x_{k-1} \quad \text{and} \quad \Delta y_k = y_k - y_{k-1}$$

we would like to define

$$\int_C f(x, y) dx = \lim_{\max \Delta s_k \rightarrow 0} f(x_k^*, y_k^*) \Delta x_k \quad \text{and} \quad \int_C f(x, y) dy = \lim_{\max \Delta s_k \rightarrow 0} f(x_k^*, y_k^*) \Delta y_k.$$

Unlike Δs_k , the values of Δx_k and Δy_k change sign if the order of the partition points along C is reversed.

In order to define the line integrals using dx or dy , we must restrict ourselves to oriented curves C and to partitions of C in which the partition points are ordered in the direction of the curve.

With this restriction, if the limits exist and do not depend on the choice of partition or sampling points, then we refer them as the line integrals of f with respect to x and y along C .

28.4.1 Evaluating line integrals with respect to x, y , and z

The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t :

$$x = x(t), \quad y = y(t), \implies dx = x'(t)dt, \quad dy = y'(t)dt.$$

Therefore,

$$\int_C f(x, y) dx = \int_a^b f(x(t), y(t)) x'(t) dt$$

and

$$\int_C f(x, y) dy = \int_a^b f(x(t), y(t)) y'(t) dt$$

Sometimes we write

$$\int_C P(x, y) dx + \int_C Q(x, y) dy = \int_C P(x, y) dx + Q(x, y) dy.$$

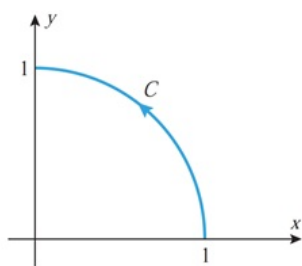


Figure 28.11: circular arc C give by $x = \cos t$, $y = \sin t$ where $0 \leq t \leq \pi/2$.

Example 28.6. Evaluate $\int_C 2xy dx + (x^2 + y^2) dy$ along the circular arc C give by

$$x = \cos t, \quad y = \sin t \quad 0 \leq t \leq \pi/2.$$

Solution. We have

$$\begin{aligned} \int_C 2xy dx &= \int_0^{\pi/2} (2 \cos t \sin t) \left[\frac{d}{dt}(\cos t) \right] dt \\ &= -2 \int_0^{\pi/2} \sin^2 t \cos t dt = -\frac{2}{3} \sin^3 t \Big|_0^{\pi/2} = -\frac{2}{3}. \end{aligned}$$

$$\begin{aligned} \int_C (x^2 + y^2) dy &= \int_0^{\pi/2} (\cos^2 t + \sin^2 t) \left[\frac{d}{dt}(\sin t) \right] dt \\ &= \int_0^{\pi/2} \cos t dt = \sin t \Big|_0^{\pi/2} = 1. \end{aligned}$$

Thus,

$$\int_C 2xy dx + (x^2 + y^2) dy = \int_C 2xy dx + \int_C (x^2 + y^2) dy = -\frac{2}{3} + 1 = \frac{1}{3}$$

□

Theorem 28.8 (Properties of line integrals).

1. $\int_{-C} f(x, y) dx = - \int_C f(x, y) dx$
2. $\int_{-C} g(x, y) dy = - \int_C g(x, y) dy$
3. $\int_{-C} f(x, y) ds = - \int_C f(x, y) ds$

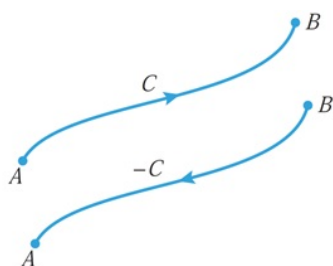


Figure 28.12: Oriented curves

$$4. \int_C f(x, y) dx + g(x, y) dy = \int_C f(x, y) dx + \int_C g(x, y) dy$$

$$5. \int_{-C} f(x, y) dx + g(x, y) dy = - \int_C f(x, y) dx + g(x, y) dy$$